3. SPEKTOR A.A., Certain three-dimensional static contact problems of the theory of elasticity with slippage and adhesion. Izv. Akad. Nauk SSSR, MTT, 3, 1981.
4. FRIDMAN V.M. and CHERNINA V.S., An iterative process for solving the finite-dimensional contact problem. Zhurn. vychisl. matematiki i mat. fiziki, 7, 1, 1967.
5. FADDEYEV D.K. and FADDEYEVA V.N., Numerical Methods of Linear Algebra. Moscow, Fizmatgiz, 1963.

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NEUTRAL LOADING IN THE ENDOCHRONIC MODEL OF THE THEORY OF PLASTICITY*

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The properties of a simple model of the endochronic theory of plasticity (ETP) for inactive deformation processes are studied. The theory uses "internal time" as the parameter of the deformation process, and no distinction is made between the loading and unloading which is essentially non-linear /1–4/.

A loading and deformation processes are described in a five-dimensional stress deviator space Σ_5 /5/. Restricting ourselves to processes not depending explicitly on time, we can write the defining equations of ETP for an initially isotropic material in the form

$$de = de_p + de_e,\ de_e = E^{-1}d\sigma,\ de_p = \sigma dz,\ dz = F(\sigma, e, e', \xi)\, d\xi, \tag{1}$$
$$d\xi = |de - \chi E^{-1}d\sigma|,\ 0 \leqslant \chi \leqslant 1$$

Here e, e_p, e_e are the total, plastic and elastic deformation vectors, respectively, σ is the stress vector, z is the "internal time" parameter, E is the modulus of elasticity, F is the O (5)-invariant hardending function, /4/, ξ is the modified measure of the deformation, and χ is a parameter characterizing the contribution of the elastic component of the deformation towards the variation in intrinsic time; for any $a, a' = da/ds$, where $ds = |de|$ is the length of the arc of the deformation trajectory. We assume that the magnitude of the volumetric deformation ε does not affect the plasticity.

System (1) yields the following equation:

$$d\sigma = Ede - EF\sigma d\xi \tag{2}$$

In the simplest case F can be regarded as a constant of the material, in which case Eq. (2) will take the form ($\sigma_0 = 1/F$ is the yield point)

$$d\sigma = Ede - \alpha\sigma\, |de - \chi E^{-1}d\sigma|,\ \alpha = E/\sigma_0 \tag{3}$$

The above equation describes a material without hardening. For a material with linear isotropic hardening we must make the substitution $\sigma_0 \to \sigma_0(1 + k\xi)$ /1/ (in the case of translational hardening $\sigma \to \sigma - \mu e$, μ is the hardening modulus).

The ETP has no concept of a yield surface and there is no unloading condition; therefore Eqs.(2), (3) are assumed to hold for both the loading (active processes $\sigma de > 0$) and unloading (passive processes $\sigma de < 0$ /6/).

Let us write the condition of inactivity of the process: $\sigma de \leqslant 0$ (the equality corresponds to neutral loading). Substituting de into it from (2), we can rewrite this condition in the form

$$\sigma_u' + EF\sigma_u\xi' \leqslant 0,\ \sigma_u = |\sigma| \tag{4}$$

Let γ be the angle between σ and $d\sigma$, $a = |\sigma'|$. Then

$$\sigma_u' = a\cos\gamma \tag{5}$$

Let us transform the expression for ξ'. To do this, we substitute de from (2) into the expression for ξ' (1) and square the resulting expression. After cancelling like terms, we obtain a quadratic equation for ξ', whose root is

$$\xi' = \frac{1-\chi}{E}\, a\, \frac{F\sigma_u\cos\gamma + \sqrt{1 - F^2\sigma_u^2\sin^2\gamma}}{1 - F^2\sigma_u^2} \tag{6}$$

(the second root is rejected as extraneous). Substituting (6) into (4) we write, after reduction, the condition of inactivity of the process when an extra load $d\sigma$ is added to the

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load σ , in the form

$$\cos\gamma \leqslant -(1-\chi)\,\varphi\,[1+(1-2\chi)\,\varphi^2]^{-1/2},\quad \varphi = F\sigma_u \tag{7}$$

The equality corresponds to the limit angle γ_* of neutral loading. In the special case of $F = 1/\sigma_0$ (model (3)) we have

$$\cos\gamma_*(\sigma_u) = -(1-\chi)\frac{\sigma_u}{\sigma_0}\left[1+(1-2\chi)\frac{\sigma_u^2}{\sigma_0^2}\right]^{-1/2} \tag{8}$$

($\gamma_* \to \arccos(-{}^1/_2\sqrt{1-\chi})$ as $\sigma_u \to \sigma_0$).

Thus even when the deformation is carried out in the "fully plastic" region $\sigma_u \simeq \sigma_0$, the condition $d\sigma_u < 0$ is insufficient for the onset of unloading (a passive process). Eq.(8) (Eq.(7) in the general case) determines $\sigma_u < \sigma_0$ in a sphere (in Σ_5), after the cones with axis σ and aperture angle $2(\pi - \gamma_*(\sigma_u))$, which may be called the unloading cones since when the load increment $d\sigma$ is applied, the unloading will begin at the point A corresponding to the stress σ, only if $d\sigma$ lies within the corresponding unloading cone. Therefore the boundary between loading and unloading (neutral loading curve) (NLC) has an angle point at A. The possibility of such behaviour of the NLC was pointed out in /6/.

We shall restrict ourselves to two-dimensional deformation-loading processes within the framework of model (3). Let us introduce, in the two-dimensional stress space, the polar coordinates $r = \sigma_u, \beta$, and measure the angle β from the initial direction of σ_i (Fig.1).

The NCL equation in these coordinates has the form

$$|d\sigma|/d\beta = \sigma_u / \sin\gamma_*(\sigma_u)$$

Using Eqs.(5) we rewrite this equation thus:

$$d\beta = \operatorname{tg}\gamma_*(\sigma_u)\,\sigma_u^{-1}\,d\sigma_u$$

Substituting expression (8) into it and integrating, we obtain

$$\beta = \pm\frac{\chi}{1-\chi}\left(\arcsin\left(\chi\frac{\sigma_u}{\sigma_0}\right) + \left(\frac{\sigma_0^2}{\chi^2\sigma_u^2} - 1\right)^{1/2}\right) + C \tag{9}$$

The integration constant C is found from the condition $\beta(\sigma_i) = 0,\ \sigma_i = |\sigma_i|$.

The relation connecting β and σ_u at $\chi = 0$ is particularly simple

$$\beta = \pm\sigma_0\,(1/\sigma_u - 1/\sigma_i) \tag{10}$$

We see from (9), (10) that the NLC is "wound on the origin of coordinates", (i.e. $\sigma_u \to 0$ as $\beta \to \infty$). The dotted line in Fig.1 shows a branch of the NLC (at $\chi = 0.9$) emerging from the point A. Its second branch is situated symmetrically about the σ_2. The origin of coordinates serves as the focus of the NLC. The tangent to the NLC at A is inclined to the σ_1 axis at the angle $\approx 9{,}1°$.

All this shows clearly that the NLC of the form shown corresponds to a material of infinitely low elasticity. We can bring in a finite region of elasticity by changing that defining equations somewhat.

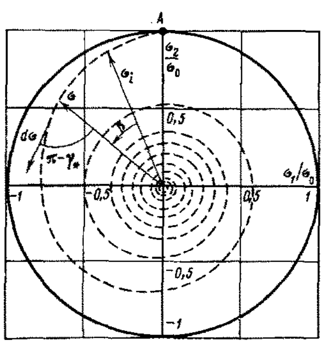

Fig.1

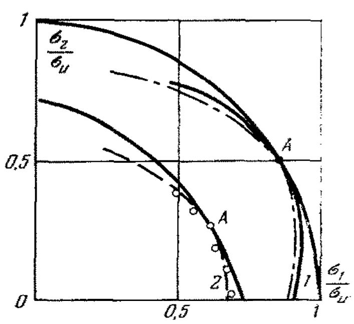

Fig.2

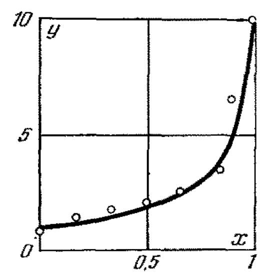

Fig.3

Let us consider (instead of (3)), the following equation ($h(x)$ is the Heaviside function, $\sigma_* < \sigma_0$):

$$d\sigma = E\,de - \frac{E}{\sigma_0 - \sigma_*}\left(1 - \frac{\sigma_*}{\sigma_u}\right) h(\sigma_u - \sigma_*)\,\sigma\,d\xi \tag{11}$$

It can be confirmed that Eq.(11) describes a material without hardening, with the limit of plasticity σ_0 and of elasticity σ_*. The sphere $\sigma_u = \sigma_*$ plays the part of the classical yield surface (we have within it $d\sigma = E de$). The NLC equation for the model (11) has the form

$$d\beta = \pm \frac{\sigma_0 - \sigma_*}{\sigma_u - \sigma_*} \left(1 - \chi^2 \frac{\sigma_u^2}{\sigma_0^2}\right)^{1/2} (1-\chi)^{-1} d\sigma_u, \quad \sigma_u > \sigma_*$$

It is clear that $\sigma_u \to \sigma_*$ as $\beta \to \infty$, i.e. the NLC is "wound" on a circle $\sigma_u = \sigma_*$, which plays the role of the limit cycle.

We note that no work is done in deforming along the NLC $dA = \sigma de = 0$.

From Eq.(8) it follows that $\gamma_* > \pi/2$, therefore the NLC originating in σ_i lies fully within the sphere $\sigma_u = \sigma_i$, and the deformation along $\sigma_u = \sigma_i$ corresponds to an active process and leads to an increase in plastic deformations.

Figure 2 shows the NLC obtained experimentally in /7/ (the solid line 1) and in /8/ (points 2), and the dot-dashes and dashes denote the curves computed from formula (9) for $\chi = 0.95$ (for case 1) and $\chi = 0.91$ (for case 2).

When $\sigma_u = \sigma_i$ is constant, we obtain from (3), (6) (remembering that $\cos\gamma = 0$)

$$de_p = \frac{1-\chi}{E} \left(1 - \frac{\sigma_u^2}{\sigma_0^2}\right)^{-1/2} \frac{\sigma}{\sigma_0} |d\sigma|$$

Thus the increment $|de_p|$ is proportional to the length of the arc of the load trajectory $ds_\sigma = |d\sigma|$ in the space Σ_5. This was observed experimentally in /9, 10/. Fig.3 shows a graph depicting the dependence of $y = \mathrm{tg}\,\psi/\mathrm{tg}\,\psi_0$, where $\mathrm{tg}\,\psi = |de_p|/ds_\sigma$ on the magnitude of the dimensionless stress $x = \sigma_u/\sigma_0$ for $\chi = 0.9$ (the solid line). The points show the results obtained experimentally /9/.

REFERENCES

1. VALANIS K.C., A theory of viscoplasticity without a yield surface. Arch. Mech. Stosow., 23, 4, 1971.
2. VALANIS K.C., On the foundations of the endochronic theory of viscoplasticity. Arch. Mech. Stosow., 27, 5-6, 1975.
3. VALANIS K.C., Fundamental consequences of a new intrinsic time measure, Plasticity as a limit of the endochronic theory. ARch. Mech., 32, 2, 1980.
4. BAZHANT Z., Endochronic theory of non-elasticity and the incremental theory of plasticity. In book: Mekhanika deformiruemykh tverdykh tel. Napravleniya razvitiya. Moscow, Mir, 1983.
5. IL'YUSHIN A.A., Plasticity. Fundamentals of a General Mathematical Theory. Moscow, Izd-vo Akad. Nauk SSSR, 1963.
6. IL'YUSHIN A.A. and LENSKII V.S., Strength of Materials. Moscow, Fizmatgiz, 1959.
7. KORENEVA A.M. and DOSHCHINSKII G.A., Experimental study of the neutral loading curve. Izv. Tom. politekhn. in-ta, 183, 1968.
8. ZHUKOV A.M., Certain special features of the neutral loading curve. Izv. Akad. Nauk SSSR OTN, 8, 1958.
9. KORENEVA A.M. and DOSHCHINSKII G.A., Experimental study of the plastic deformation of "neutral loading". Izv. Tom. politekhn. in-ta, 183, 1968.
10. DOSHCHINSKII G.A. and POGOZHEVA N.V., On the principles of growth of plastic deformation under a complex load of constant intensity with a fixed direction of the principal axes. Izv. Tom. politekhn. in-ta, 241, 1975.

Translated by L.K.